LAMINAR HEAT TRANSFER OVER THE INITIAL SECTION OF A RECTANGULAR CHANNEL

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The author describes a study, based on the linearized energy equation, of heat transfer over the initial section of a rectangular channel with a laminar incompressible flow and both uniform and nonuniform temperature fields at the inlet. Temperature and Nusselt number distributions over the channel perimeter are given.

It has been shown [1] that in the case of laminar flow it is not permissible to apply the resistance law derived for circular tubes to other cross sections simply by substituting the hydraulic for the ordinary radius. For example, the resistance coefficient for a circular tube is given by

$$\Psi = 16/\mathrm{Re},$$

whereas for a rectangular one it is

$$\Psi = C/\mathrm{Re},$$

where C = 14.225 for a square section, and C = 24 for a flat channel. This is corroborated by the data of reference [2], in which the ratio of the sides of the rectangle is shown to affect the friction coefficient in laminar flow. It is thus clear that in laminar flow the analogy between heat and momentum transfer is not a basis for adapting data on the Nu number for a circular tube to a rectangular channel.

It should be noted that laminar flow over the initial section of a tube has been observed even at Reynolds numbers greater than critical. It was shown in [2] and [3], for example, that there is a region of laminar flow, corresponding to a dip in the pressure curve, in the initial section of a rectangular channel with $\mathrm{Re} > \mathrm{Re}_{cr}$. The presence of a region of laminar heat transfer in the initial section of a tube has been noted experimentally in a number of papers, e.g., in [4, 5, 6], where laminar flow occurred right up to $\mathrm{Re} = 10^5$. The theoretical study of laminar heat transfer in the initial section of a rectangular channel has great practical significance.

Since the temperature field at the inlet to the individual channels in regenerators and compact heat exchangers is nonuniform, there is considerable interest in an examination of the influence of nonuniformity of the temperature field at the inlet on heat transfer conditions in the initial section of a channel and on the temperature distribution over its perimeter.

The energy equation of the boundary layer for a steady flow of incompressible fluid with constant physical properties and without energy dissipation has the form [7]:

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = \lambda \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right). \tag{1}$$

In this equation let us compare the convective terms $u \frac{\partial T}{\partial x}$, $v \frac{\partial T}{\partial y}$, $w \frac{\partial T}{\partial z}$.

Since the flow near each of the walls in the initial section of a rectangular channel is similar to that over an infinite flat plate, to compare the convective terms in the energy equation, we may use the solution of the Blasius problem for the velocity distribution in flow over a flat plate, which also holds for the temperature field when Pr = 1. It was shown in [8] that the ratio $\left. v \frac{\partial T}{\partial y} \Big/ u \frac{\partial T}{\partial x} \right|_{\mathrm{Pr}=1}$ has an almost constant value, equal to 0.5, in the part of the boundary where the longitudinal velocity component changes from u = 0 to u = 0.95; only at a considerable distance from the wall, practically outside the limits of the boundary layer, does it begin to vary appreciably.

Thus, we may write

$$v \frac{\partial T}{\partial y} \Big/ u \frac{\partial T}{\partial x} = \mathrm{const}, \quad w \frac{\partial T}{\partial z} \Big/ u \frac{\partial T}{\partial x} = \mathrm{const},$$

and the energy equation for three-dimensional flow in the boundary layer takes the form

$$\varepsilon' u \frac{\partial T}{\partial x} = a \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right), \tag{2}$$

where

$$\varepsilon' = 1 + \left(v \frac{\partial T}{\partial y} \bigg/ u \frac{\partial T}{\partial x}\right) + \left(w \frac{\partial T}{\partial z} \bigg/ u \frac{\partial T}{\partial x}\right).$$

At the inlet to the channel the velocity is constant over the section and equal to U; moreover, it has been shown in [9] that replacing the longitudinal velocity component u by its mean value over the section U leads to only small changes in the solution of the equation, which can be taken into account by introducing into the equation a correction ε. We then have

$$\varepsilon U \frac{\partial T}{\partial x} = a \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{3}$$

The correction ε was evaluated in [8]:

$$\varepsilon = 0.346 \, \mathrm{Pr}^{-\frac{1}{3}}.$$

Equation (3) is the equation of unsteady heat conduction, and methods of solving it are widely known [10, 11].

Introducing the dimensionless variables $\xi = x/h$, $\eta = y/h$, $\zeta = z/h$ where h is half the distance between the walls of the rectangular channel perpendicular to the y axis, and the dimensionless temperature $\theta = (T - T_0)/T_0$ (T_0 is the characteristic temperature of the fluid at the inlet section), we obtain

$$\varepsilon \mathrm{Pe} \frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\partial^2 \theta}{\partial \zeta^2}. \tag{4}$$

In the case of given heat flux through the walls of the rectangular channel, the boundary conditions may be written:

$$\begin{gathered}
\xi = 0, \quad \theta = \beta_y \eta + \gamma_y \eta^2 + \beta_z \zeta + \gamma_z \zeta^2, \\
\eta = \pm 1, \quad \frac{\partial \theta}{\partial \eta} = \pm K_{0y} \pm K_{1y} \xi, \\
\zeta = \pm \zeta_0, \quad \frac{\partial \theta}{\partial \zeta} = \pm K_{0z} \pm K_{1z} \xi.
\end{gathered} \tag{5}$$

Here $\zeta_0 = z_0/h$ is the distance from the x axis to the wall in the direction of the z axis.

The solution of (4) for initial and boundary conditions (5) may be represented in the form of a sum of the solutions of the two following equations for the temperature distribution in an infinite flat channel:

$$\varepsilon \mathrm{Pe} \frac{\partial \theta_y}{\partial \xi} = \frac{\partial^2 \theta_y}{\partial \eta^2}, \quad \begin{gathered} \xi = 0, \ \theta_y = \beta_y \eta + \gamma_y \eta^2, \\ \eta = \pm 1, \ \frac{\partial \theta_y}{\partial \eta} = \pm K_{0y} \pm K_{1y} \xi; \end{gathered} \tag{6}$$

$$\varepsilon \mathrm{Pe} \frac{\partial \theta_z}{\partial \xi} = \frac{\partial^2 \theta_z}{\partial \zeta^2}, \quad \begin{gathered} \xi = 0, \ \theta_z = \beta_z \zeta + \gamma_z \zeta^2, \\ \zeta = \pm \zeta_0, \ \frac{\partial \theta_z}{\partial \zeta} = \pm K_{0z} \pm K_{1z} \xi. \end{gathered} \tag{7}$$

It is easily seen that the sum of the solutions of (6) and (7) is a solution of (4) that satisfies the boundary conditions (5):

$$\theta = \theta_y + \theta_z.$$

Solutions of (6) were obtained in [12], and applying these to the initial section of the channel ($\xi \le 100$) for $\mathrm{Re} \ge 3 \cdot 10^3$, we obtain

$$\theta = 2(K_{0y} - 2\gamma_y - \beta_y)\sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}}\ \mathrm{i\,erfc}\left(\frac{1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + 8K_{1y}\xi\sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}}\ \mathrm{i^3\,erfc}\left(\frac{1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + \beta_y\eta + \gamma_y\eta^2 + 2(K_{0z} - 2\gamma_z\zeta_0 - \beta_z)\sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}}\ \mathrm{i\,erfc}\left(\frac{\zeta_0-\zeta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + 8K_{1z}\xi\sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}}\ \mathrm{i^3\,erfc}\left(\frac{\zeta_0-\zeta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + \beta_z\zeta + \gamma_z\zeta^2. \quad (8)$$

Knowing the temperature distribution inside the channel, we can find the mean temperature of the fluid and the temperature of the wall and from these calculate the Nu number. Finally, we obtain

$$\mathrm{Nu}|_{\zeta=\zeta_0} = A_1/B_1,\quad A_1 = K_{0z} + K_{1z}\xi,$$

$$B_1 = 2(K_{0y} - 2\gamma_y - \beta_y)\sqrt{\xi/\varepsilon \mathrm{Pe}}\ \mathrm{i\,erfc}\left(\frac{1-\eta}{2}\sqrt{\varepsilon \mathrm{Pe}/\xi}\right) + \beta_y\eta + \gamma_y\eta^2 + 8K_{1y}\xi\sqrt{\xi/\varepsilon \mathrm{Pe}}\ \mathrm{i^3\,erfc}\left(\frac{1-\eta}{2}\sqrt{\varepsilon \mathrm{Pe}/\xi}\right) + 1.1284(K_{0z} - 2\gamma_z\zeta_0 - \beta_z)\sqrt{\xi/\varepsilon \mathrm{Pe}} + 0{,}752K_{1z}\xi\sqrt{\xi/\varepsilon \mathrm{Pe}} + \beta_z\zeta_0 + \gamma_z\zeta_0^2 - \left[\frac{K_{0z}\xi + K_{1z}\xi^2}{\zeta_0\varepsilon \mathrm{Pe}} + \frac{\gamma_z\zeta_0^2}{3} + \frac{K_{0y}\xi + K_{1y}\xi^2}{\varepsilon \mathrm{Pe}} + \frac{\gamma_y}{3}\right]; \quad (9)$$

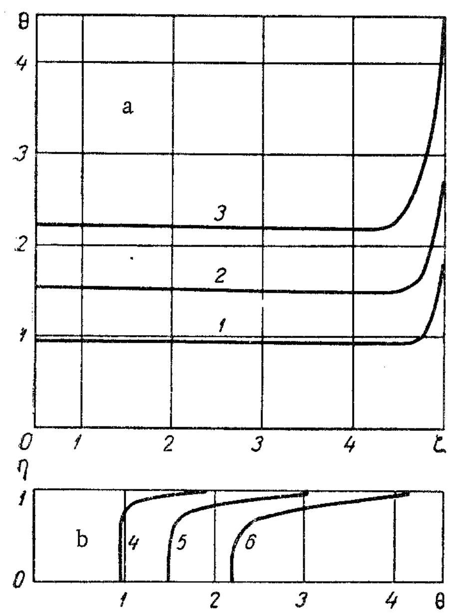

Fig. 1. Temperature distribution over the perimeter of a rectangular channel $z_0/h = 5$ (a – over the long side, b – over the short side) for uniform temperature field at the inlet and given heat flux, $\mathrm{Re} = 10^4$, $\mathrm{Pr} = 0.7$: 1, 4) $\xi = 20$; 2, 5) 50; 3, 6) 100.

$$\mathrm{Nu}|_{\eta=1} = A_2/B_2,\quad A_2 = K_{0y} + K_{1y}\xi,$$

$$B_2 = 1.1284(K_{0y} - 2\gamma_y - \beta_y)\sqrt{\xi/\varepsilon \mathrm{Pe}} + 0.752K_{1y}\xi\sqrt{\xi/\varepsilon \mathrm{Pe}} + \beta_y + \gamma_y + 2(K_{0z} - 2\gamma_z\zeta_0 - \beta_z)\sqrt{\xi/\varepsilon \mathrm{Pe}} \times \mathrm{i\,erfc}\left(\frac{\zeta_0-\zeta}{2}\sqrt{\varepsilon \mathrm{Pe}/\xi}\right) + 8K_{1z}\xi\sqrt{\xi/\varepsilon \mathrm{Pe}}\,\mathrm{i^3erfc} \times \left(\frac{\zeta_0-\zeta}{2}\sqrt{\varepsilon \mathrm{Pe}/\xi}\right) + \beta_z\zeta + \gamma_z\zeta^2 - \left[\frac{K_0\xi + K_1\xi^2}{\zeta_0\varepsilon \mathrm{Pe}} + \frac{\gamma_z\zeta_0^2}{3} + \frac{K_{0y}\xi + K_{1y}\xi^2}{\varepsilon \mathrm{Pe}} + \frac{\gamma_y}{3}\right]. \quad (10)$$

For a uniform temperature field at the inlet and constant heat flux over the perimeter and along the length of the channel ($K_{0y} = K_{0z} = K_0$, $K_{1y} = K_{1z} = 0$), we have

$$\mathrm{Nu}|_{\eta=1} = \frac{1}{D},$$

$$D = 1{,}1284\sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}} + 2\sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}}\ \mathrm{i\,erfc}\left(\frac{\zeta_0-\zeta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) - \frac{\xi(1+\zeta_0)}{\xi_0\varepsilon \mathrm{Pe}}.$$

The formula for $\mathrm{Nu}|_{\zeta=\zeta_0}$ is obtained by replacing $\zeta_0 - \zeta$ by $1 - \eta$.

Figure 1 gives the temperature distribution over the perimeter of a rectangular channel at various sections for a uniform temperature field at the inlet. The channel section is a rectangle with a ratio of the sides 1 : 5.

It is seen from the graph that a temperature increase occurs in the channel corner, the width of the region of increased temperature being identical for both the short and the long sides of the rectangle. Therefore, the greater the ratio of the sides of a rectangular channel, the greater the portion of the short side occupied by a region of increased temperature.

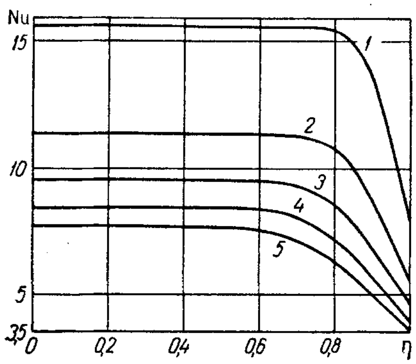

Fig. 2. Nu number distribution over the short side of a rectangular channel with $z_0/h = 5$ for uniform temperature field at the inlet and given heat flux, Re = 10^4, Pr = 0.7: 1) $\xi = 10$, 2) 20, 3) 30, 4) 40, 5) 50.

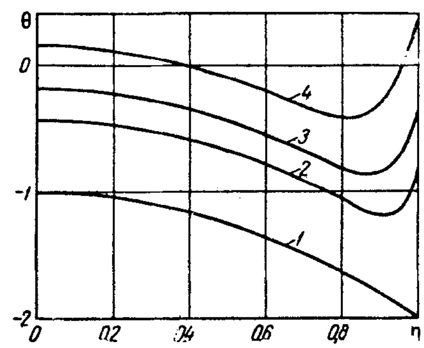

Fig. 3. Temperature distribution over the perimeter of a square channel with nonuniform temperature field at the inlet and given heat flux, Re = 10^4, Pr = 0.7, $K_0 = 10$, $\gamma = -1$: 1) $\xi = 0$, 2) 5, 3) 10, 4) 20.

Figure 2 shows the Nu number distribution over the short side of a rectangular channel at various distances from the inlet. It is seen that there is a considerable reduction in the Nu number in the corner.

A nonuniform temperature field at the inlet leads to a rather complicated temperature distribution over the perimeter of a rectangular channel and, in particular, of a square channel (Fig. 3), where the temperature increase is much stronger in the corner than in the middle of the channel.

If the temperature field at the channel inlet is such that the temperature of the layers nearest the wall is lower than that of the fluid in the middle of the channel, then the mean mass temperature of the fluid near the inlet is higher than the wall temperature. At certain points on the channel section, with increasing distance from the inlet, the mean fluid temperature becomes equal to the wall temperature at that point, which corresponds to Nu = $\pm\infty$. Figure 4 gives the Nu number distribution over the side of a square at channel sections lying at different distances from the inlet. Examination of the curves shows that at $\xi = 5$ there is one break in the Nu number distribution, corresponding to the point $\eta = 0.45$, while at a distance $\xi = 10$ from the inlet Nu = $\pm\infty$ at two points: $\eta = 0.65$ and $\eta = 0.97$.

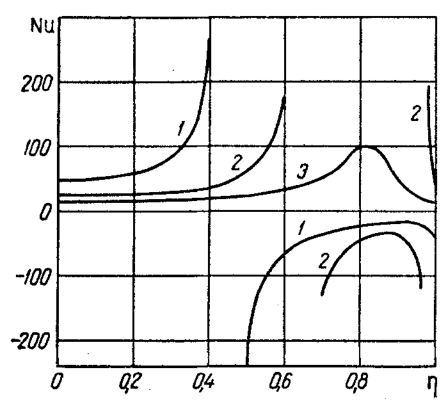

Fig. 4. Distribution of Nu number over the perimeter of a square channel for nonuniform temperature field at the inlet and given heat flux, Re = 10^4, Pr = 0.7, K_0 = = 10, $\gamma = -1$: 1) $\xi = 5$, 2) 10, 3) 20.

For a given constant wall temperature Eq. (4) must be solved with the following boundary and initial conditions:

$$\left.\begin{matrix}\eta = \pm 1 \\ \zeta = \pm\zeta_0\end{matrix}\right\} \theta = 0,$$

$$\xi = 0,\ \theta = \theta_y\theta_z = (1-\beta_y\eta - \gamma_y\eta^2)(1-\beta_z\zeta - \gamma_z\zeta^2). \tag{11}$$

Here $\theta = (T_W - T)/(T_W - T_0)$.

The product of the solutions of the two following equations $\theta = \theta_y\theta_z$ is a solution of (4) with conditions (11):

$$\varepsilon\,\mathrm{Pe}\,\frac{\partial\theta_y}{\partial\xi} = \frac{\partial^2\theta_y}{\partial\eta^2},\qquad \begin{matrix}\eta = \pm 1,\ \theta_y = 0, \\ \xi = 0,\ \theta_y = 1 - \beta_y\eta - \gamma_y\eta^2;\end{matrix} \tag{12}$$

$$\varepsilon\,\mathrm{Pe}\,\frac{\partial\theta_z}{\partial\xi} = \frac{\partial^2\theta_z}{\partial\zeta^2},\qquad \begin{matrix}\zeta = \pm\zeta_0,\ \theta_z = 0, \\ \xi = 0,\ \theta_z = 1 - \beta_z\zeta - \gamma_z\zeta^2.\end{matrix} \tag{13}$$

Using the solution of (12) obtained for $\mathrm{Re} \geq 3 \cdot 10^3$ and $\xi \leq 100$, we get:

$$\theta = \left[\frac{8\gamma_y\xi}{\varepsilon\,\mathrm{Pe}}\, i^2\,\mathrm{erfc}\left(\frac{1-\eta}{2}\sqrt{\frac{\varepsilon\,\mathrm{Pe}}{\xi}}\right) + (\gamma_y+\beta_y-1)\,\mathrm{erfc}\left(\frac{1-\eta}{2}\sqrt{\frac{\varepsilon\,\mathrm{Pe}}{\xi}}\right) + \right.$$

$$\left. + 1-\beta_y\eta-\gamma_y\eta^2-\frac{2\gamma_y\xi}{\varepsilon\,\mathrm{Pe}}\right]\left[\frac{8\gamma_z\xi}{\varepsilon\,\mathrm{Pe}}\, i^2\,\mathrm{erfc}\left(\frac{\zeta_0-\zeta}{2}\sqrt{\frac{\varepsilon\,\mathrm{Pe}}{\xi}}\right)+(\gamma_z\zeta_0^2+\beta_z\zeta_0-\right.$$

$$\left. -1)\,\mathrm{erfc}\left(\frac{\zeta_0-\zeta}{2}\sqrt{\frac{\varepsilon\,\mathrm{Pe}}{\xi}}\right)+1-\beta_z\zeta-\gamma_z\zeta^2-\frac{2\gamma_z\xi}{\varepsilon\,\mathrm{Pe}}\right]; \tag{14}$$

$$\mathrm{Nu}\,|_{\eta=1} = \frac{A_1}{B}, \quad A_1 = \left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1},$$

$$\mathrm{Nu}\,|_{\zeta=\zeta_0} = \frac{A_2}{B}, \quad A_2 = \left.\frac{\partial\theta}{\partial\zeta}\right|_{\zeta=\zeta_0},$$

$$B = \left[\frac{8}{3}\,\frac{\gamma_z\xi^{3/2}}{\zeta_0\sqrt{\pi}(\varepsilon\,\mathrm{Pe})^{3/2}}+\frac{2\,(\gamma_z\zeta_0^2-1)}{\zeta_0}\sqrt{\frac{\xi}{\pi\varepsilon\,\mathrm{Pe}}}+1-\frac{\gamma_z\zeta_0^2}{3}-\right.$$

$$\left.-\frac{2\gamma_z\xi}{\varepsilon\,\mathrm{Pe}}\right]\left[\frac{8}{3}\,\frac{\gamma_y}{\sqrt{\pi}}\left(\frac{\xi}{\varepsilon\,\mathrm{Pe}}\right)^{3/2}+2\,(\gamma_y-1)\sqrt{\frac{\xi}{\varepsilon\,\mathrm{Pe}}}+1-\frac{\gamma_y}{3}-\frac{2\gamma_y\xi}{\varepsilon\,\mathrm{Pe}}\right]. \tag{15}$$

NOTATION

x – longitudinal coordinate; y, z – transverse coordinates; u – longitudinal velocity component; v, w – transverse velocity components; U – fluid velocity at channel inlet section; T – fluid temperature; α – thermal diffusivity; λ – thermal conductivity; 2h, $2z_0$ – height and width of rectangular channel; Re, Pe, Pr – Reynolds, Peclet, and Prandtl numbers.

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